The Interaction of Particles in the Gravitational Fields of Mini Black Holes

Lauren Greenspan, New York University

Advisor: Dr. A.M. Mukhamedzhanov, Texas A&M University

Goals:

*To find solutions to the Klein-Gordon (K-G) equation to describe particle interaction around mini black holes (BH)

*To calculate the absorption cross-section of nuclei by the mini BH

Curved Space

The generalized metric:

$$ds^2 = g_{00}(\mathbf{r},t)c^2dt^2 - 2g_{0i}cdtdx^i - g_{ji}(\mathbf{r},t)dx^j dx^i,$$

$$i,j = 1,2,3$$

describes time (t) and space in three dimensions (r).

In General Relativity, observation depends on the adopted coordinate system. For static black holes, we generally consider the Schwarzschild metric, which has a nonphysical singularity at the event horizon: $r = r_s$

Where the Schwarzschild radius

$$r_s = \frac{2GM}{c^2}$$

This singularity means that a particle will never reach the event horizon!

To remove the Schwarzschild Singularity we introduce Eddington-Finkelstein (E-F) coordinates with the contravariant and covariant tensors:

$$g^{\mu\nu} = \begin{pmatrix} 1+\frac{r_s}{r} & -\frac{r_s}{r} & 0 & 0 \\ -\frac{r_s}{r} & -(1-\frac{r_s}{r}) & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix} \qquad g_{\mu\nu} = \begin{pmatrix} 1-\frac{r_s}{r} & -\frac{r_s}{r} & 0 & 0 \\ -\frac{r_s}{r} & -(1+\frac{r_s}{r}) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$$g_{\mu\nu}g^{\mu\nu} = I$$

$$g = \det g_{\mu\nu} = -r^4\sin^2\theta < 0$$

The negative determinant indicates that locally, the space is pseudo-Euclidean.

Evaluating the general metric with elements of the contravariant tensor, we get the E-F metric:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = (1-\frac{r_s}{r})c^2dt^2 - 2\frac{r_s c}{r}dt\,dr - (1+\frac{r_s}{r})dr^2 - r^2 d\Omega$$

Black Holes

*Star-collapse- Formed by catastrophic collapse of a massive star ($M>M_{sun}$)

*Primordial (mini)- Formed by density irregularities just after the Big Bang

*Galactic-merger of mini BH's???

Mini Black Holes

To use this QM approach, the wavelength of a particle in the field of the mini BH must be comparable to its r_s. We consider a BH with r_s on the order of fm (to compare-the sun would form a BH of about 3km) and around 10^{15}g (19 orders less massive than the sun). BH's of this size would have a lifetime around the age of the Universe. Mini BH's would violently evaporate the particles they've gradually accumulated in an amount of Hawking radiation inversely proportional to the mass of the BH. In this model, particles interact quantum mechanically in the relativistic space-time of a mini BH, making it a possible setting to study exciting new physics like quantum gravity.

The Klein-Gordon (KG) Equation

The extension of the KG equation for curved space is

$$\hbar^2 \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\sqrt{-g}\,g^{\mu\nu}\frac{\partial}{\partial x^\nu}\Psi = -m^2c^2\Psi$$

leading to

$$\hbar^2\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial x^0}\left(r^2\sin\theta g^{00}\frac{\partial}{\partial x^0}\psi\right) + \frac{\partial}{\partial x^0}\left(r^2\sin\theta g^{01}\frac{\partial}{\partial x^1}\psi\right) + \ldots\right] = -m^2c^2\psi$$

$$\psi \equiv \psi(t,r,\theta,\varphi) \qquad \psi(t,\mathbf{r},\theta,\varphi) = e^{i\frac{E}{\hbar}t}\psi(\mathbf{r},\theta,\varphi)$$

Eventually, we get the radial Schrödinger equation (SE) describing the total motion of a particle in the gravitational field of the BH:

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2} + \hat{V}_1(r) + \hat{V}_2(r) + V^N(r) + \frac{l(l+1)}{2\mu r^2}\right]U_l(k,r) = E_p U_l(k,r)$$

where, l is the orbital angular momentum of the particle.

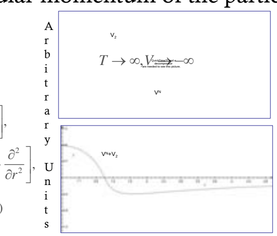
$$\psi_l(k,r) = \frac{U_l(k,r)}{kr},$$

$$V^N = -\frac{GMm}{r},$$

$$\hat{V}_1(r) = -i(\hbar c)r_s\left[-\frac{1}{2r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right],$$

$$\hat{V}_2(r) = \frac{1}{2}\frac{(\hbar c)^2}{mc^2}\frac{r_s}{r}\left[\frac{1}{r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\right],$$

$$\hat{T}_r\psi_l(r) = -\frac{\hbar^2}{2m}\frac{1}{kr}\frac{\partial^2}{\partial r^2}U_l(k,r)$$



Absorption Cross-Section

Using a standard procedure, we get the particle conservation law:

$$-i\hbar\int d\vec{r}\,\nabla\vec{j} = -i\hbar\int d\vec{s}\bullet\vec{j} = \int d\vec{r}\left(\psi^* V_1\psi - \psi V_1^*\psi^*\right)$$

$$\vec{j} = -i\frac{\hbar}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*)$$

The LHS is proportional to the absorption cross-section.

So, we can use the V_1 term in the cross section equation to figure Out the absorption.

$$\sigma = i\frac{2m}{\hbar^2k}\int d\vec{r}\left(\psi^*V_1\psi - \psi V_1^*\psi^*\right) = i\frac{2m}{\hbar^2k}2\int d\vec{r}\psi^* V_1\psi$$

$$= i\frac{4m}{\hbar^2k}(4\pi)^2\sum_{l=0}^{\infty}\frac{2l+1}{4\pi}\int_{r>r_s}^{\infty}dr\cdot r^2\frac{U_l(k,r)}{kr}\left(-i\hbar\left[\frac{cr_s}{r}\frac{\partial}{\partial r} + \frac{r_sc}{2r^2}\right]\frac{U_l(k,r)}{kr}\right)$$

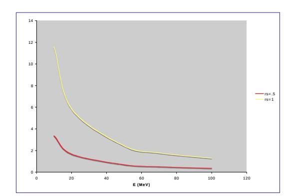
$$= 8\pi\frac{mc^2}{\hbar c}\frac{1}{k^3}\sum_{l=0}^{\infty}(2l+1)U_l^2(k,r)$$

$$U_l(k,r) = e^{-\frac{\pi\eta}{2}}\frac{|\Gamma(l+1+i\eta)|}{2\Gamma(2l+2)}(2kr)^{l+1}e^{-ikr}{}_1F_1(l+1-i\eta,2l+2,2ikr)$$

where Γ is a Gamma function, ${}_1F_1$ is a hypergeometric function obtained for V^N, and V_1+V_2 are taken to be a perturbation.

Using an analogy to the Coulomb potential, we get:

$$\eta^c = \frac{-z_1z_2e^2m}{\hbar k} \to \eta = \frac{-GMm^2}{\hbar k} = \frac{-r_s}{\frac{\hbar}{mc}\sqrt{\frac{2E}{mc^2}}}, \qquad k = \frac{mv}{\hbar}$$



Mass gained by a mini BH going through the Sun

$\rho = 1\frac{g}{cm^3}$ Sun density

$m = 1.66\cdot10^{-24}g$ proton mass

$N = \frac{1}{1.66\cdot10^{-23}} = 6.024\cdot10^{23}cm^{-3}$ volume per proton

$\lambda = \frac{1}{N\sigma} = \frac{1}{6.02\cdot10^{23}*100\cdot10^{-24}}cm = 0.0166cm$ mean free path

$for\ \sigma = 100b$

$R = 7\cdot10^{10}cm$

$N = \frac{2R}{\lambda} = 8.44\cdot10^{12}$

$M = Nm = 1.4\cdot10^{-11}g$

R is the radius of the sun, N is the number of collisions experienced by mini BH, and M is the gained mass, which is negligible!